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Schur function analysis of the unitary discrete series representations of the non-compact symplectic group

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Abstract. We develop and analyse certain character reductions for the infinite-dimensional unitary discrete series representations of the non-compact symplectic group $\mathrm{Sp}(2n,\mathbb{R})$. The group reductions considered are $\mathrm{Sp}(2k,\mathbb{R}) \supset \mathrm{Sp}(2,\mathbb{R}) \times \mathrm{O}(k)$ and the more general $\mathrm{Sp}(2nk,\mathbb{R}) \supset \mathrm{Sp}(2n,\mathbb{R}) \times \mathrm{O}(k)$. We use Schur function techniques to derive succinct formulae involving certain infinite series of Schur functions. The results are relevant to the study of many-body systems with interactions of bilinear form and to the description of various quantum phenomena including collective behaviour.

1. Introduction

In this paper we present some character theory results of the non-compact group $\mathrm{Sp}(2n,\mathbb{R})$, in particular analysing the unitary discrete series irreducible representations (uds-irreps) for the group reductions $\mathrm{Sp}(2nk,\mathbb{R}) \supset \mathrm{Sp}(2n,\mathbb{R}) \times \mathrm{O}(k)$ and, in particular, $\mathrm{Sp}(2k,\mathbb{R}) \supset Sp(2,\mathbb{R}) \times \mathrm{O}(k)$. We use the powerful Schur function techniques which have been widely used in the compact group case (see King 1975, Black *et al* 1983). These have the advantage of giving rank-independent results. We elaborate on the work of King and Wybourne (1985), whom we follow closely in presenting much of the character theory of the uds-irreps. These representations of $\mathrm{Sp}(2n,\mathbb{R})$ in general have been encountered in other areas, in particular in the study of the harmonic oscillator, the hydrogen atom and the theory of nuclear collective motion.

The interest is that we wish to apply these results to the many-body microscopic system of N electrons confined in d dimensions and to describe its collective behaviour (see Haase and Johnson 1992). In recent years such systems have been grown artificially and are known as quantum devices. The confinement in such quantum devices is often electrostatic in origin and the effective confining potential is, to a good approximation, quadratic (see Kumar *et al* 1990). Model Hamiltonians of such quantum systems, even in the presence of a uniform magnetic field, can be taken to be composed of bilinear operators of the Heisenberg algebra, and hence the quantum behaviour is described by the non-compact symplectic group $\mathrm{Sp}(2Nd,\mathbb{R})$ and its unitary representations. A full discussion of the collective behaviour of the N-particle system in a d-dimensional space requires the various group–subgroup reductions of the representations of $\mathrm{Sp}(2Nd,\mathbb{R})$ to determine and classify basis states of the system, and ultimately the matrix elements of the generators of $\mathrm{Sp}(2Nd,\mathbb{R})$ and its subgroups. In the field of nuclear physics, similar work has

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been performed by Moshinsky, Kramer and Rowe and their respective coworkers (see Castaños *et al* 1984, Moshinsky *et al* 1985, Kramer 1982, Rowe and Rosensteel 1979, Rosensteel and Rowe 1983, and references therein) focusing, in particular, on group chains involving $\mathrm{Sp}(2n,\mathbb{R}) \supset U(n)$. Our focus is on those chains involving $\mathrm{Sp}(2nk,\mathbb{R}) \supset \mathrm{Sp}(2n,\mathbb{R}) \times \mathrm{O}(k)$.

We separate the paper into three sections: the first outlines the partition notation used, the Schur function operations and Schur function series required, and gives briefly those compact group results we require later; the second discusses the two basic uds-spin irreps of $\mathrm{Sp}(2n,\mathbb{R})$ giving their various properties, the general compact group reduction $\mathrm{Sp}(2n,\mathbb{R}) \supset U(n)$, and the $\mathrm{Sp}(2k,\mathbb{R}) \supset \mathrm{Sp}(2,\mathbb{R}) \times \mathrm{O}(k)$ reduction of the uds-irreps; the third presents an analysis of the $\mathrm{Sp}(2nk,\mathbb{R}) \supset \mathrm{Sp}(2n,\mathbb{R}) \times \mathrm{O}(k)$ reduction of the uds-irreps using two group–subgroup schemes, one involving the $\mathrm{Sp}(2n,\mathbb{R}) \supset U(n)$ reduction while the other involves the $\mathrm{Sp}(2k,\mathbb{R}) \supset \mathrm{Sp}(2,\mathbb{R}) \times \mathrm{O}(k)$ reduction. The derivation of these results are rank-independent wherein lies the power of the Schur function techniques. Although the final analysis is incomplete in that a Schur function formula cannot be easily obtained, we outline an inversion technique which yields leading lowest-weight terms to any order required. We re-establish known results for particular cases. We are also in the process of evaluating computer-generated results for particular cases to provide some insight into simplifications.

2. The reductions of the compact group

For compact groups, irreps are denoted by partitions and their properties, such as branching rules and Kronecker products, are given by certain operations of Schur functions which describe the characters of the irreps. We briefly discuss the notation to be used and give a quick treatment of some compact group character results below. For more details see King (1975) and Black *et al* (1983).

From the representation theory of finite and compact groups, every finite-dimensional non-equivalent irrep can be denoted by a partition (μ)—an ordered set of positive integers $\mu_1 \geqslant \cdots \geqslant \mu_k > 0$. The integers μ_i are called the parts of the partition while the weight of the partition is the sum of its parts $w(\mu) = \mu_1 + \cdots + \mu_k$. The partition $(\tilde{\mu})$ is called the conjugate partition of (μ) formed by interchanging columns and rows of the associated Young diagram. The number of parts of (μ) is $\tilde{\mu}_1$.

We use the convention of distinguishing characters and irreps of the orthogonal group $\mathrm{O}(n)$ by square brackets [..], those of the symplectic group $\mathrm{Sp}(2n)$ by angular brackets $\langle .. \rangle$ and those of the unitary group $\mathrm{U}(n)$ by curly brackets $\{..\}$. For clarity, we shall often subscript the rank of the group, here n, after the brackets.

For completeness and to help introduce many of the Schur function techniques we require, we give the relevant compact group reductions. As we are dealing only with the covariant tensor irreps of $\mathrm{U}(n)$, the standard labelling is by a single partition $\{\mu\}$ into, at most, n parts. The $\mathrm{U}(mn) \supset \mathrm{U}(m) \times \mathrm{U}(n)$ reduction is for such irreps given by

$$\{\mu\}_{mn} \downarrow \sum_{\xi} \{\mu \circ \xi\}_m \times \{\xi\}_n \qquad (1)$$

where the symbol $\circ$ signifies the Schur function operation of inner multiplication, known more in association with Kronecker products of symmetric group irreps. The summation over partitions (ξ) is restricted to, at most, n parts. An example is

$$\{21\}_{mn}\downarrow\{21\}_m\times\{3\}_n+\{3\}_m\times\{21\}_n+\{21\}_m\times\{21\}_n+\{1^3\}_m\times\{21\}_n+\{21\}_m\times\{1^3\}_n \tag{2}$$

where we have used the inner products $(21\circ 3)=(21)$, $(21\circ 21)=(3+21+1^3)$, and $(21\circ 1^3)=(21)$. The above result is valid for all m and n but if the number of parts of the partition exceed the rank of the group then the character is null.

The $\mathrm{U}(n)\supset\mathrm{O}(n)$ reduction for covariant irreps is given by

$$\{\mu\}_n\downarrow[\mu/D]_n=\sum_\delta[\mu/\delta]_n \tag{3}$$

where / denotes the Schur function operation of division, and D represents an infinite sum of partitions (δ) whose parts δ_i are all even,

$$D=(0)+(2)+(4)+(2^2)+(6)+(42)+(2^3)+\cdots. \tag{4}$$

We note that, although D is formally an infinite series, the division operation renders the summation finite by restricting the terms to partitions no greater than the partition (μ). For example

$$\{32\}_n\downarrow[32/D]_n=[32/(0+2+2^2)]_n=[32+3+21+1]_n. \tag{5}$$

Another result we need is the character reduction for $\mathrm{O}(mn)\supset\mathrm{O}(m)\times\mathrm{O}(n)$

$$[\mu]_{mn}\downarrow\sum_\zeta[((\mu/C)\circ\zeta)/D]_m\times[\zeta/D]_n \tag{6}$$

where C is the inverse series of D, $CD=1$, with leading terms

$$C=(0)-(2)+(31)-(41^2)-(3^2)+(431)+(51)+\cdots. \tag{7}$$

The summation is over all partitions (ζ) into min(m,n) parts and of even (odd) weight if $w(\mu)$ is even (odd) and no greater than $w(\mu)$. This is due to (i) the nature of the C series where the weights are all even; (ii) the property of division where $w(\mu/\nu)=w(\mu)-w(\nu)$; and (iii) the property of inner multiplication which is always between partitions of the same weight. Unravelling the content of the reduction formula is best illustrated:

$$\begin{aligned}[21]_{mn}\downarrow\sum_\zeta[((21/(0-2))\circ\zeta)/D]_m\times[\zeta/D]_n] &= \sum_\zeta[((21-1)\circ\zeta)/D]_m\times[\zeta/D]_n\\ &= [(21\circ 3)/D]_m\times[3/D]_n+[(21\circ 21)/D]_m\times[21/D]_n\\ &\quad+[(21\circ 1^3)/D]_m\times[1^3/D]_n-[(1\circ 1)/D]_m\times[1/D]_n\\ &= [21/D]_m\times[3/D]_n+[(3+21+1^3)/D]_m\times[21/D]_n\\ &\quad+[21/D]_m\times[111/D]_n-[1/D]_m\times[1/D]_n\end{aligned}$$

where now $[3/D]=[3+1]$, $[21/D]=[21+1]$, $[111/D]=[111]$ and $[1/D]=[1]$. Applying these results yields the general reduction valid for any m and n

$$\begin{aligned}[21]_{mn}\downarrow\,&[21+1]_m\times[3+1]_n+[3+1+21+1+111]_m\times[21+1]_n\\ &+[21+1]_m\times[111]_n-[1]_m\times[1]_n\\ \downarrow\,&[21+1]_m\times[3]_n+[3+21+111+(2)1]_m\times[21]_n\\ &+[21+1]_m\times[111]_n+[3+(2)21+111+(2)1]_m\times[1]_n\end{aligned}$$

where the multiplicities of [21] × [1], [1] × [21] and [1] × [1] are placed in parentheses. Note the cancellation of the negative term. In general this will always happen despite the fact that the C series contains negative terms.

The results represented in these examples are valid for all m and n, however to apply these to particular values the partitions may not be standard irrep labels of their respective orthogonal group, that is of $\mathrm{O}(m)$ or $\mathrm{O}(n)$, and need to be modified accordingly. This may lead to further cancellations. A standard $\mathrm{O}(n)$ irrep character label is one in which the partition $[\mu]$ has at most $\lfloor n/2 \rfloor$ parts otherwise it is non-standard. If $[\mu]$ is non-standard, $\tilde{\mu}_1 > \lfloor n/2 \rfloor$, then one must apply the $\mathrm{O}(n)$ modification rule. This rule removes a hook length of length $h = 2\tilde{\mu}_1 - n$, starting at the bottom of the first column of the associated Young diagram and removing h contiguous boxes along the boundary of the diagram. If the resulting diagram, denoted symbolically as $[\mu - h]$, is a regular partition then $[\mu] = (-1)^{c-1}[(\mu - h)^*]$ where c is the column in which the removal procedure ends and $*$ represents the associated irrep character. Otherwise if the resulting partition is irregular, it is the null character ϕ. If one begins with the associated character $[\mu^*]$ then $[\mu^*] = (-1)^{c-1}[\mu - h]$. The hook length removal procedure must be repeated until a standard irrep label or the null label is obtained. By way of illustration if $[\mu] = [3321]$ and $h = 3, 4, 5$, then $[\mu - h] = -[33^*], \phi, [31^*]$ respectively.

In describing many of the properties of $\mathrm{Sp}(2n, \mathbb{R})$ uds-irreps in the next section, we shall need the following definition. We shall call a label $[\kappa]$ of $\mathrm{O}(k)$ *near*-standard if $\tilde{\kappa}_1 + \tilde{\kappa}_2 \leqslant k$ for the reason that if (κ) is non-standard then it requires only one application of the modification rule to arrive at a standard label. One must remove the hook length $h = 2\tilde{\kappa}_1 - k$, which, because $\tilde{\kappa}_1 - h = k - \tilde{\kappa}_1 \geqslant \tilde{\kappa}_2$, is always taken completely from the first column ($c = 1$) and always yields a regular partition so that we have $[\kappa] = [(\kappa - h)^*]$.

3. The unitary discrete series representations of $\mathrm{Sp}(2n, \mathbb{R})$

We follow King and Wybourne (1985) in presenting much of the representation theory of $\mathrm{Sp}(2n, \mathbb{R})$. For the sake of brevity, we shall denote the non-compact symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ by $\mathrm{Sp}(2n)$. The latter is often used for the compact symplectic group $\mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(n)$ which is not considered here at all. There exist two basic uds-irreps, $\langle s;0_+\rangle$ and $\langle s;0_-\rangle$, of $\mathrm{Sp}(2n)$; both are infinite dimensional and non-faithful. Moreover, there exists the spin representation $\langle s;0\rangle$ which is a faithful unitary irrep of the double covering group, the so-called metaplectic group $\mathrm{Mp}(2n)$ of $\mathrm{Sp}(2n)$, and which is reducible into a sum of the two basic spin irreps of $\mathrm{Sp}(2n)$

$$\langle s;0\rangle_n \downarrow \langle s;0_+\rangle_n + \langle s;0_-\rangle_n . \tag{8}$$

There are two fundamental results associated with the basic spin irrep $\langle s;0\rangle_n$. The first refers to its reduction as a sum of irrep characters of the maximal compact group $\mathrm{U}(n)$,

$$\langle s;0\rangle_n \downarrow {\epsilon_n}^{1/2} \bullet M = {\epsilon_n}^{1/2} \bullet \sum_m \{m\}_n \tag{9}$$

where the summation is over all non-negative integers, $\epsilon_n \equiv \{1^n\}$ denotes the one-dimensional determinantal representation of $\mathrm{U}(n)$, and $\{m\}$ is a covariant tensor irrep of $\mathrm{U}(n)$.

The second result is that the reduction of the basic spin irrep $\langle s;0\rangle$ of $Sp(2nk)$ to $Sp(2n)\times O(k)$ is

$$\langle s;0\rangle_{nk}\downarrow\sum_{\kappa}\langle\tfrac{1}{2}k(\kappa)\rangle_n\times[\kappa]_k \tag{10}$$

where the summation is over all the partitions (κ) satisfying the constraints $\tilde{\kappa}_1\leqslant n$ and $\tilde{\kappa}_1+\tilde{\kappa}_2\leqslant k$ (Kashiwara and Verge 1978, Rowe *et al* 1985). Note that the branching is multiplicity-free and that there is for each irrep label of $Sp(2n)$ just one irrep label for $O(k)$ and *vice versa*—a fact known in the nuclear physics field as complementarity (Moshinsky and Quesne 1970). The constraints imply that the summation is over those standard partitions that label covariant tensor irreps of $U(n)$, and those irrep labels of $O(k)$ that are *near*-standard since $\tilde{\kappa}_1$ could be greater than $k/2$ but is certainly less than k.

It is just the two results given by equations (9) and (10) that are needed to determine properties of the uds-irreps $\langle\frac{1}{2}k(\kappa)\rangle$ of $Sp(2n)$. For example, the branching rule appropriate for the reduction $Sp(2n)\supset U(n)$ was shown to take the form (King and Wybourne 1985)

$$\langle\tfrac{1}{2}k(\kappa)\rangle_n\downarrow\epsilon_n{}^{k/2}\bullet\{({}_s\kappa^k\bullet D)_K\}_n\qquad\text{with } K=\min(k,n) \tag{11}$$

where we first note that $\langle\frac{1}{2}k(\kappa)\rangle_n$ is a unitary irrep therefore (κ) is a near-standard label of $O(k)$, and second ${}_s(\kappa)^k$ is called the signed sequence of (κ) which involves a sum of partitions $\pm(\nu)$ such that $\pm[\nu]_k$ is equivalent to $[\kappa]_k$ under the modification rule of $O(k)$. As this rule removes hook lengths, one can see that this sum is infinite, being the inverse procedure of repeated appending of hook lengths. For example

$${}_s(54)^4=(54)-(542)+(5431)-(543^2)-(54^21^2)+\cdots. \tag{12}$$

The outer multiplication $({}_s\kappa^k\bullet D)_K$ can be thought of as being carried out in the group $U(K)$ as implied by the subscript K. This fact imposes certain limits on the partitions appearing in each of the infinite sequences D and ${}_s(\kappa)^k$. The former remains infinite but is restricted to partitions into at most K parts while the latter is rendered finite with at most $1+K-\tilde{\kappa}_1$ terms. The reason is that each added hook length must begin in the first column. The above example gives ${}_s(54)^4=(54)-(542)$ with $K=3$. More details on the signed sequence ${}_s(\kappa)^k$ can be found in Rowe *et al* (1985) and King and Wybourne (1985). One further remark on this reduction is that the leading terms as determined by the smallest $U(n)$ irrep can be seen to be

$$\langle\tfrac{1}{2}k(\kappa)\rangle_n\downarrow\epsilon_n{}^{k/2}\bullet(\{\kappa\}_n+\{\kappa\bullet 2\}_n+\cdots) \tag{13}$$

where the first term gives a $U(n)$-justification for the labelling of the $Sp(2n)$ irreps.

A second result that follows from the properties of the basic spin irreps but not given in King and Wybourne (1985) is the general branching rule for $Sp(2n)\supset Sp(2)\times O(n)$ which is summarized in the following:

$$\langle\tfrac{1}{2}k(\kappa)\rangle_n\downarrow\sum_m\langle\tfrac{1}{2}nk(m)\rangle_1\times[((m/C)\circ({}_s\kappa^k\bullet D)_k)_n/D]_n. \tag{14}$$

Athough more complex, there is some simplification in the evaluation of the expression $((m/C)\circ({}_s\kappa^k\bullet D)_k)_n$. First, as (m) is only a 1-part partition, $(m/C)=(m)-(m-2)$ for $m\geqslant 2$, special cases $(0/C)=(0)$, $(1/C)=(1)$.

Second, ${}_s(\kappa)^k$ with partitions restricted to at most k parts becomes a finite sum. Third, the inner products are particularly easy as for m integer $(m)\circ(\nu) = (\nu)$ where (ν) is any partition of weight m. Hence for given m the purpose of the inner product is to extract from $({}_s\kappa^k \bullet D)_k$ those terms of weight m and $m-2$. The terms arising from the inner products must also be standard labels of U(n) hence the subscript n in $((m/C)\circ({}_s\kappa^k \bullet D)_k)_n$. Note, as the inner products are especially simple, we can take the part restrictions inside the inner products. As a consequence, with $K = \min(k,n)$, we could write

$$(m/C)\circ({}_s\kappa^k \bullet D)_K = ({}_s\kappa^k \bullet D)_K^m - ({}_s\kappa^k \bullet D)_K^{m-2} \tag{15}$$

where after the parentheses the superscript determines the weight restriction and the subscript the combined part restriction. Also, if (κ) is a partition of even (odd) weight, the weights of the partitions appearing in $({}_s\kappa^k \bullet D)$ are also even (odd) and therefore m must be accordingly an even (odd) integer. Letting $w(\kappa) = w$, the leading terms in the reduction as determined by the smallest-labelled Sp(2) irreps can be seen to be

$$\langle\tfrac{1}{2}k(\kappa)\rangle_n \downarrow \langle\tfrac{1}{2}nk(w)\rangle_1 \times [\kappa/D]_n + \langle\tfrac{1}{2}nk(w+2)\rangle_1 \times [(\kappa\bullet 2-\kappa)_K/D]_n + \cdots \tag{16}$$

with the proviso that if (κ') is the second term in $({}_s\kappa^k)$ then $w(\kappa')\rangle w + 2$. If $w(\kappa') = w + 2$ then (κ') must be subtracted from the product $(\kappa \bullet 2)$. Note that the first term gives perhaps an alternative O(n)-justification for the labelling of the Sp($2n$) irreps.

To exemplify the reduction formula given by equation (14) or (16) we present the leading smallest-weight terms for the reduction of the Sp(6) irrep $\langle 2(32)\rangle_3$ to Sp(2) × O(3). We need the outer product sequence for (32), ${}_s(32)_3^4 = (32) - (32^2)$ with the D series restricted to, at most, three parts, yielding

$${}_s(32)_3^4 \bullet D_3 = (32) + (52) + (43) + (421) + (3^21) + \cdots. \tag{17}$$

The Schur function division by the D series then gives

$$\langle 2(32)\rangle_3 \downarrow \langle 6(5)\rangle_1 \times [3+2^*+1]_3 + \langle 6(7)\rangle_1 \times [5+(2)4^*+(3)3+(3)2+1+0^*]_3 + \cdots \tag{18}$$

where we have used O(3) modifications.

As the reduction Sp($2n$) ⊃ Sp(2) × O(n) has not been previously analysed, it is instructive to derive equation (14) in detail as a example of the Schur function technique. Starting with the metaplectic irrep $\langle s;0\rangle_{nk}$, we employ two different chains of groups:

(A) Sp($2nk,\mathbb{R}$) ⊃ Sp($2n,\mathbb{R}$) × O(k) ⊃ Sp($2,\mathbb{R}$) × O(n) × O(k)

$$\langle s;0\rangle_{nk} \downarrow \sum_{\kappa}\langle\tfrac{1}{2}k(\kappa)\rangle_n \times [\kappa]_k$$

$$\downarrow \sum_{\kappa\lambda\mu} \mathrm{m}(\kappa,\lambda\mu)\langle\tfrac{1}{2}nk(\lambda)\rangle_1 \times [\mu]_n \times [\kappa]_k$$

which essentially defines the problem, that of finding $\mathrm{m}(\kappa,\lambda\mu)$, with (κ) and (λ) near-standard in O(k) and O(1) respectively but (μ) standard in O(n); and

(B) $\mathrm{Sp}(2nk,\mathbb{R}) \supset \mathrm{Sp}(2,\mathbb{R}) \times \mathrm{O}(nk) \supset \mathrm{Sp}(2,\mathbb{R}) \times \mathrm{O}(n) \times \mathrm{O}(k)$

$$\begin{aligned}\langle s;0\rangle_{nk} &\downarrow \sum_{l} \langle \tfrac{1}{2}nk(l)\rangle_1 \times [l]_{nk} \\ &\downarrow \sum_{l\zeta} \langle \tfrac{1}{2}nk(l)\rangle_1 \times [((l/C)\circ(\zeta)_k)_n/D]_n \times [\zeta/D]_k \\ &= \sum_{l\xi} \langle \tfrac{1}{2}nk(l)\rangle_1 \times [((l/C)\circ(\xi\bullet D)_k)_n/D]_n \times [\xi]_k\end{aligned}$$

where l is a non-negative integer and (ζ), (ξ) are partitions of weight determined by terms in (l/C) and restricted to k parts. Comparing the two final results from (A) and (B) yields $(\lambda) = (l)$, $(\xi) \subset_s (\kappa)^k$, and for a given (κ) and (l)

$$\sum_{\mu} m(\kappa, l\mu)(\mu)_n = (((l/C)\circ({}_s\kappa^k \bullet D)_k)_n/D)_n. \tag{19}$$

On substituting this into (A) we obtain the general reduction of the uds-irrep under $\mathrm{Sp}(2n) \supset \mathrm{Sp}(2) \times \mathrm{O}(n)$ in a very compact form as given in equation (14).

4. The Schur function analysis of the $\mathrm{Sp}(2nk) \supset \mathrm{Sp}(2n) \times \mathrm{O}(k)$ reduction

We analyse in this section the more general reduction $\mathrm{Sp}(2nk) \supset \mathrm{Sp}(2n) \times \mathrm{O}(k)$ using the above Schur function techniques. The analysis of this reduction is incomplete in that a succinct formula cannot be obtained due to the complex dependence on the values n, k and the interdependence of the Schur function operations. However, a procedure given by King and Wybourne (1985) is used by which a step-by-step operation extracts the leading terms in lowest-weight partitions to any order required. This is based on the fact that the uds-irreps are labelled by the highest weight of the lowest $\mathrm{U}(n)$ constitutent irrep as can be seen in equation (13).

Let us consider the reduction of the irrep $\langle \frac{1}{2}m(\mu)\rangle_{nk}$ of $\mathrm{Sp}(2nk)$ under the two chains of groups, (A) and (B), given below. We note that, as we are dealing only with the unitary irreps, decompositions to symplectic subgroups will involve only unitary irreps:

(A) $\mathrm{Sp}(2nk) \supset \mathrm{Sp}(2n) \times \mathrm{O}(k) \supset \mathrm{U}(n) \times \mathrm{O}(k)$. Using equation (11) we have

$$\begin{aligned}\langle \tfrac{1}{2}m(\mu)\rangle_{nk} &\downarrow \sum_{\nu\kappa} \mathrm{m}(\mu\langle \tfrac{1}{2}mk(\nu)\rangle_n \times [\kappa]_k \\ &\downarrow \sum_{\nu\kappa} \mathrm{m}(\mu,\nu\kappa)\epsilon_n{}^{mk/2} \bullet \{({}_s\nu^{mk}\bullet D)_N\}_n \times [\kappa]_k\end{aligned}$$

where (κ) is a near-standard label of $\mathrm{O}(k)$, hence $\tilde{\kappa}_1 + \tilde{\kappa}_2 \leqslant k$, (ν) is a standard label of $\mathrm{Sp}(2n)$ with $\tilde{\nu}_1 \leqslant n$ and $\tilde{\nu}_1 + \tilde{\nu}_2 \leqslant mk$, and $N = \min(mk, n)$.

(B) $\mathrm{Sp}(2nk) \supset \mathrm{U}(nk) \supset \mathrm{U}(n) \times \mathrm{U}(k) \supset \mathrm{U}(n) \times \mathrm{O}(k)$. Using equations (11), (1) and (3), we have

$$\begin{aligned}\langle \tfrac{1}{2}m(\mu)\rangle_{nk} &\downarrow \epsilon_{nk}{}^{m/2} \bullet \{({}_s\mu^m \bullet D)_M\}_{nk} \\ &\downarrow \sum_{\zeta} (\epsilon_n{}^{mk/2} \times \epsilon_k{}^{mn/2}) \bullet (\{({}_s\mu^m \bullet D)_M \circ (\zeta)_k\}_n \times \{\zeta\}_k)\end{aligned}$$

$$= \sum_{\zeta} (\epsilon_n{}^{mk/2} \bullet \{({}_s\mu^m \bullet D)_M \circ (\zeta)_k\}_n) \times (\epsilon_k{}^{mn/2} \bullet \{\zeta\}_k)$$

$$\downarrow \sum_{\zeta} (\epsilon_n{}^{mk/2} \bullet \{({}_s\mu^m \bullet D)_M \circ (\zeta)_k\}_n) \times (\epsilon_k{}^{mn/2} \bullet [\zeta/D]_k)$$

$$= \sum_{\xi} (\epsilon_n{}^{mk/2} \bullet \{({}_s\mu^m \bullet D)_M \circ (\xi \bullet D)_k\}_n) \times (\epsilon_k{}^{mn/2} \bullet [\xi]_k)$$

where $M = \min(m,nk)$. There are several remarks to make here. The summations over (ζ) and (ξ) involve partitions into at most k parts. The terms in the outer product $({}_s\mu^m \bullet D)_M$ must be restricted to at most $M = \min(m, nk)$ parts, and hence ${}_s(\mu)^m$ and D can be restricted to partitions into at most M parts. The inner product selects from (ζ) and $(\xi \bullet D)$ partitions with the same weight as those in the expression $({}_s\mu^m \bullet D)$. Moreover, in the inner product itself, $({}_s\mu^m \bullet D)_M \circ (\xi \bullet D)_k$, only terms of at most n parts are retained. Note that the k and n part restrictions cannot be taken inside the inner product as these products are quite non-trivial.

Comparing the final results of (A) and (B), we have for a given (μ) and (κ)

$$\sum_{\nu} m\{\mu, \nu\kappa\}\{({}_s\nu^{mk} \bullet D)_N\}_n = \{({}_s\mu^m \bullet D)_M \circ ({}_s\kappa^k \bullet D)_k\}_n \quad (20)$$

where the inner product implies that $|w(\mu) - w(\kappa)|$ is always an even integer. Multiplying by $\epsilon_n{}^{mk/2}$ yields on the left-hand side a sum of uds-irrep characters of $\mathrm{Sp}(2n)$

$$\sum_{\nu} m(\mu, \nu\kappa)\langle \tfrac{1}{2}mk(\nu)\rangle_n \downarrow \ \epsilon_n{}^{mk/2} \bullet \{({}_s\mu^m \bullet D)_M \circ ({}_s\kappa^k \bullet D)_k\}_n \quad (21)$$

expanded as a sum of $\mathrm{U}(n)$ irrep characters. The problem now is to identify on the right-hand side of this equation each signed sequence $({}_s\nu^{mk} \bullet D)_N$ and hence (ν). One can use the property of inner multiplication

$$(\kappa \bullet \lambda) \circ \mu = \sum_{\rho} (\kappa \circ (\mu/\rho)) \bullet (\lambda \circ \rho) \quad (22)$$

to rearrange the right-hand side to get

$$\{({}_s\mu^m \bullet D)_M \circ ({}_s\kappa^k \bullet D)_k\}_n = \sum_{\rho} \{(({}_s\kappa^k) \circ (({}_s\mu^m \bullet D)_M/\rho)) \bullet (D \circ \rho)\}_n \quad (23)$$

where the sum is over all partitions (ρ) which are of even weight. This equation is not quite in the required form of equation (11) that it can be inverted. In some special cases the inversion is possible but, in general, the identification is not easy to show algebraically due to the complexity of the expression. However, for a specific case $\langle \tfrac{1}{2}m(\mu)\rangle_{nk}$, one can tediously arrive at a leading lowest-weight expansion by commencing with lowest-weight partitions (κ), expanding the right-hand side expression of equation (20) in leading lowest-weight terms for each (κ), and recursively identifying the signed sequences of the partition (ν) by identifying the lowest-weight partition and then subtracting its signed sequence from the expression, thus leaving the next lowest-weight term. In this way one can resolve the identification and invert equation (21). This leading lowest-weight expansion can be extended by including more partitions (κ).

As a check though of the validity of this result we evaluate the reduction of the basic spin irreps of $Sp(2nk)$:

(1) $\langle\frac{1}{2}m(\mu)\rangle = \langle\frac{1}{2}(0)\rangle$ giving $M = 1$, and $({}_s(0)^1 \bullet D)_1 = D_1$ which contains even 1-part partitions (that is, all even integers), hence with $K = \min(k, n)$

$$\begin{aligned}\{({}_s0^1 \bullet D)_1 \circ ({}_s\kappa^k \bullet D)_k\}_n &= \{(D_1) \circ ({}_s\kappa^k \bullet D)_k\}_n \\ &= \{({}_s\kappa^k \bullet D)_k^{\text{even}}\}_n = \{({}_s\kappa^k \bullet D)_K^{\text{even}}\}_n\end{aligned}$$

where the inner product restricts ${}_s(\kappa)^k$ and hence (κ) to partitions of even weight. The last result is in the appropriate form that, when multiplied by ${\epsilon_n}^{k/2}$, it gives the unitary character $\langle\frac{1}{2}k(\kappa)\rangle_n$ of $Sp(2n)$ when reduced to $U(n)$. Therefore we arrive at the result

$$\langle\tfrac{1}{2}(0)\rangle_{nk} \downarrow \sum_{\kappa\,\text{even}} \langle\tfrac{1}{2}k(\kappa)\rangle_n \times [\kappa]_k \tag{24}$$

summing over even-weight partitions with the constraint $\tilde{\kappa}_1 \leqslant n$ and $\tilde{\kappa}_1 + \tilde{\kappa}_2 \leqslant k$.

(2) $\langle\frac{1}{2}m(\mu)\rangle = \langle\frac{1}{2}(1)\rangle$ giving $M = 1$ and $({}_s(1)^1 \bullet D)_1 = (1 \bullet D)_1$ which contains only odd 1-part partitions (that is all odd integers), hence with $K = \min(k, n)$

$$\begin{aligned}\{({}_s1^1 \bullet D)_1 \circ ({}_s\kappa^k \bullet D)_k\}_n &= \{(1 \bullet D)_1 \circ ({}_s\kappa^k \bullet D)_k\}_n \\ &= \{({}_s\kappa^k \bullet D)_k^{\text{odd}}\}_n = \{({}_s\kappa^k \bullet D)_K^{\text{odd}}\}_n\end{aligned}$$

where the inner product restricts ${}_s(\kappa)^k$ and hence (κ) to partitions of odd weight. The last result when multiplied by ${\epsilon_n}^{k/2}$ gives the unitary character $\langle\frac{1}{2}k(\kappa)\rangle_n$ of $Sp(2n)$ when reduced to $U(n)$. Therefore we arrive at the result

$$\langle\tfrac{1}{2}(1)\rangle_{nk} \downarrow \sum_{\kappa\,\text{odd}} \langle\tfrac{1}{2}k \times [\kappa]_k \tag{25}$$

summing over odd-weight partitions (κ) with the constraint $\tilde{\kappa}_1 \leqslant n$ and $\tilde{\kappa}_1 + \tilde{\kappa}_2 \leqslant k$.

It is instructive to analyse the reduction $Sp(2nk) \supset Sp(2n) \times O(k)$ employing different chains of groups other than those involving the unitary subgroups. Having obtained the $Sp(2n) \supset Sp(2) \times O(n)$ result in the previous section we can go further by repeating the process for the general uds-irrep $\langle\frac{1}{2}m(\mu)\rangle$:

(A) $Sp(2nk) \supset Sp(2n) \times O(k) \supset Sp(2) \times O(n) \times O(k)$

$$\begin{aligned}\langle\tfrac{1}{2}m(\mu)\rangle_{nk} &\downarrow \sum_{\nu\kappa} m(\mu, \nu\kappa)\langle\tfrac{1}{2}mk(\nu)\rangle_n \times [\kappa]_k \\ &\downarrow \sum_{\nu\kappa} \mathrm{m}(\mu, \nu\kappa)\langle\tfrac{1}{2}mnk(l)\rangle_1 \times [((l/C) \circ ({}_s\nu^{mk} \bullet D)_{mk})_n / D]_n \times [\kappa]_k\end{aligned}$$

(B) $Sp(2nk) \supset Sp(2) \times O(nk) \supset Sp(2) \times O(n) \times O(k)$

$$\begin{aligned}\langle\tfrac{1}{2}m(\mu)\rangle_{nk} &\downarrow \sum_{l} \langle\tfrac{1}{2}mnk(l)\rangle_1 \times [((l/C) \circ ({}_s\mu^m \bullet D)_m)_{nk} / D]_{nk} \\ &\downarrow \sum_{l\zeta} \langle\tfrac{1}{2}mnk(l)\rangle_1 \times [((((l/C) \circ ({}_s\mu^m \bullet D)_m)_{nk}) \circ (\zeta)_k) / D]_n \times [\zeta/D]_k \\ &\downarrow \sum_{l\xi} \langle\tfrac{1}{2}mnk(l)\rangle_1 \times [((((l/C) \circ ({}_s\mu^m \bullet D)_m)_{nk}) \circ (\xi \bullet D)_k) / D]_n \times [\xi]_k .\end{aligned}$$

For fixed l and (κ) we have on comparing the final result of (A) and (B)

$$\sum_{\nu} m(\mu,\nu\kappa)[((l/C)\circ({}_s\nu^{mk}\bullet D)_m k)_n/D]_n = [(((((l/C)\circ({}_s\mu^m\bullet D)_m)_{nk})\circ({}_s\kappa^k\bullet D)_k)/D]_n . \tag{26}$$

Inner product restrictions imply that terms in (l/C) and $({}_s\mu^m \bullet D)$ must have the same weight, as should those in $(((l/C)\circ({}_s\mu^m\bullet D))/C)$ and $({}_s\kappa^k\bullet D)$. One can rearrange the right hand side so that the inner product with (l/C) is outside all the parentheses and performed last. As mentioned earlier, this is because of the simple nature of this type of inner product so that we have

$$[(((((l/C)\circ({}_s\mu^m\bullet D)_m)_{nk})\circ({}_s\kappa^k\bullet D)_k)_n/D]_n = [(l/C)\circ(({}_s\mu^m\bullet D)_M\circ({}_s\kappa^k\bullet D)_k)_n/D]_n . \tag{27}$$

This relation includes the identity obtained in equation (20) and provides a consistency check of the Schur function analysis of the $\mathrm{Sp}(2nk)\supset\mathrm{Sp}(2n)\times\mathrm{O}(k)$ reduction.

5. Conclusions

The results described here demonstrate the usefulness of the Schur function techniques in studying the reduction properties of the unitary discrete series irreps of the non-compact symplectic group $\mathrm{Sp}(2n)$. We have obtained a succinct formula for the reduction $\mathrm{Sp}(2k)\supset\mathrm{Sp}(2)\times\mathrm{O}(k)$ and analysed the more complex $\mathrm{Sp}(2nk)\supset\mathrm{Sp}(2n,R)\times\mathrm{O}(k)$ reduction In the latter, we have difficulties in obtaining a succinct Schur function formula. However, we have outlined a procedure by which the leading terms can be extracted to any order. We are evaluating computer-generated results of particular cases to provide some insight into simplifications, particularly in the more difficult $\mathrm{Sp}(2nk)$ reduction. The application of this work to quantum devices is currently in progress and will be discussed elsewhere (Haase and Johnson 1992).

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